

Identity between a micrococcus-form and *bacillus*-form has already been noted.

M. Miguel, who has recently studied in a most thorough manner the germs found in the air, gives figures of the development of an organism which, at one stage of its life, has all the characters of a very long *bacillus*, and afterwards by segmentation into spherules of equal size, forms chaplets of *micrococci*, liable to separate into small groups.

The editor of the *Revue Scientifique*, that stronghold of the microbe contagion theory, admits, in a late issue, that the forms found in disease are probably varieties of habitat, and not species, yet still considers them as the cause of the diseases they accompany.

After admitting the great variability of these simple organisms, in accordance with their habitat, is it not arguing in a circle to maintain that varieties caused by certain conditions are themselves the primary cause of those conditions?

On the Reversion of Series and its Application to the Solution of Numerical Equations. By J. G. Hagen, S. J. Prof. College of the Sacred Heart, Prairie du Chien, Wisconsin.

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In a treatise entitled "Die allgemeine Umkehrung gegebener Functionen," which was published in 1849, Professor Schlömilch maintains, that all the methods of reversing series, based upon the theory of Combinations, fail in the point of *practical application* and that even Lagrange's formula presents an *unfavorable form* of such reversions. The author then proceeds to develop two new methods of reversing any given function, the one by means of Fourier's series, the other by definite integrals. In a theoretical view, Professor Schlömilch's methods are no doubt preferable to all the ancient ones on account of both their generality and their simplicity; yet when there is question about computing the numerical values of the coefficients of a reversed series, it should not be forgotten, that in most cases these definite integrals, in spite of their elegant form, can not be computed except by development, thus in many cases causing even greater trouble than the old method of combinations in the case of algebraic functions.

The treatise here published does not claim to furnish a new method, but is intended to give the recurring formula for determining the coefficients of the reversed series such a perspicuous form as to render its practical application easy, and then to apply the same to the solution of numerical equations.

PART I.

§1. Given the series $\mathfrak{X} = \sum_{r=0}^{r=m} A_r y^r$, to find the reversed series $y = \sum_{\delta=0}^{\delta=\mu} B_\delta \mathfrak{X}^\delta$,

where μ and B are unknown. Replacing y by the latter series, we obtain

$$\mathfrak{X} = \sum_{r=0}^{r=m} A_r \left[\sum_{\delta=0}^{\delta=\mu} B_\delta \mathfrak{X}^\delta \right]^r \text{ or, applying Waring's formula,}$$

$$\mathfrak{X} = \sum_{r=0}^{r=m} A_r \pi(r) \sum_{a, r} \frac{B_0^{a_0} B_1^{a_1} \dots B_\mu^{a_\mu}}{\pi(a_0) \pi(a_1) \dots \pi(a_\mu)} \mathfrak{X}^{0a_0 + 1a_1 + 2a_2 + \dots + \mu a_\mu}$$

In the formula, $\pi(r) = 1 \cdot 2 \cdot 3 \dots r$, according to the notation of *Gauss*, and the series $a_0, a_1, \dots, a_\mu$ represents all such combinations of the numbers 0, 1, 2, 3, $\dots$ r as satisfy the condition

$$a_0 + a_1 + a_2 + \dots + a_\mu = r.$$

The last formula is an identical equation and, according to the theorem of Indeterminate Coefficients, may be resolved into the following conditions:

1. Case. $0a_0 + 1a_1 + 2a_2 + \dots + \mu a_\mu = 0.$

This equation admits of the following combinations:

a_0	a_1	a_2	a_3	r
0	0	0	0	0
1	0	0	0	1
2	0	0	0	2
.	.	.	.	.

hence we have the condition

$$\sum_{r=0}^{r=m} A_r \pi(r) \frac{B_0^r}{\pi(r)} = 0 \text{ or } \sum_{r=0}^{r=m} A_r B_0^r = 0. \quad (1)$$

Though this equation is of the m th degree with regard to B_0 , yet for the reversion of series, but one of its roots is fit, because there is *but one way of developing y into a series of ascending powers of $\mathfrak{X}$* , and indeed we find that B_0 is to satisfy still another condition.

Putting $y = 0$ we have $\mathfrak{X} = A_0$, hence

$$\sum_{\delta=0}^{\delta=\mu} B_\delta A_0^\delta = 0.$$

In the special case $A_0 = 0$ we have $B_0 = 0$.

2. Case. $0a_0 + 1a_1 + 2a_2 + \dots + \mu a_\mu = 1.$

This case admits of the following combinations:

a_0	a_1	a_2	a_3	a_4	r
0	1	0	0	0	1
1	1	0	0	0	2
2	1	0	0	0	3
.	.	.	.	.	.

Hence we have the second condition

$$\sum_{r=0}^{r=m} A_r \pi(r) \frac{B_0 B_1}{\pi(r-1)} = 1. \quad (2)$$

3. *Case.* $0a_0 + 1a_1 + 2a_2 + \dots + \mu a_\mu = 2.$

This equation admits of the combinations :

a_0	a_1	a_2	a_3	a_4	r	a_0	a_1	a_2	a_3	a_4	r
0	0	1	0	0	1	0	2	0	0	0	2
1	0	1	0	0	2	1	2	0	0	0	3
2	0	1	0	0	3	2	2	0	0	0	4
.	.	.	.	.	.	.	.	.	.	.	.

Hence the third condition is

$$\sum A_r \pi(r) \left[\frac{B_0^{r-1} B_2}{\pi(r-1) \pi(1)} + \frac{B_0^{r-2} B_1^2}{\pi(r-2) \pi(2)} \right] = 0. \quad (3)$$

4. *Case.* $0a_0 + 1a_1 + 2a_2 + \dots + \mu a_\mu = 3.$

The combinations of this case are the following :

a_0	a_1	a_2	a_3	a_4	a_5	r	a_0	a_1	a_2	a_3	a_4	a_5	r	a_0	a_1	a_2	a_3	a_4	a_5	r
0	0	0	1	0	0	1	0	1	1	0	0	.	2	0	3	0	0	etc.		3
1	0	0	1	0	0	2	1	1	1	0	0	.	3	1	3	0	0			4
2	0	0	1	0	0	3	2	1	1	0	0	.	4	2	3	0	0			5
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Hence the fourth condition

$$\sum_{r=0}^{r=m} A_r \pi(r) \left[\frac{B_0^{r-1} B_r}{\pi(r-1) \pi(1)} + \frac{B_0^{r-2} B_1 B_2}{\pi(r-2) \pi(1) \pi(1)} + \frac{B_0^{r-3} B_1^3}{\pi(r-3) \pi(3)} \right] = 0. \quad (4)$$

5. *Case.* $0a_0 + 1a_1 + 2a_2 + \dots + \mu a_\mu = 4.$

Here we have the following possible combinations :

a_0	a_1	a_2	a_3	a_4	a_5	r	a_0	a_1	a_2	a_3	r	a_0	a_1	a_2	a_3	r
0	0	0	0	1	0	1	0	1	0	1	2	0	2	1	0	3
1	0	0	0	1	0	2	1	1	0	1	3	1	2	1	0	4
2	0	0	0	1	0	3	2	1	0	1	4	2	2	1	0	5
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

a_0	a_1	a_2	r	a_0	a_1	a_2	a_3	r
0	4	0	4	0	0	2	0	2
1	4	0	5	1	0	2	0	3
2	4	0	6	2	0	2	0	4
.	.	.	.	.	.	.	.	.

Hence the fifth condition :

$$\sum_{r=0}^{r=m} A_r \pi(r) \left[\frac{B_0^{r-1} B_4}{\pi(r-1) \pi(1)} + \frac{B_0^{r-2}}{\pi(r-2)} \left(\frac{B_1 B_3}{\pi(1) \pi(1)} + \frac{B_2^3}{\pi(2)} \right) + \frac{B_0^{r-3} B_1^2 B_2}{\pi(r-3) \pi(2) \pi(1)} \right]$$

$$\left. + \frac{B_0^{r-4} B_{(4)}}{\pi(r-4) \pi(4)} \right] = 0, \quad (5)$$

and so on.

Note.—The lower limit of r may be put $= 0$, because all those terms in which π contains a *negative* argument, are zero.

§2. The equations (1) to (5) may be transformed in the following way. We put for brevity's sake

$$\Sigma_\delta = \sum_{r=\delta}^{r=m} \binom{r}{\delta} A_r B_0^{r-\delta} = A_\delta + \dots, \quad (6)$$

where

$$\binom{r}{\delta} = \frac{r(r-1)(r-2)\dots(r-\delta+1)}{1 \cdot 2 \cdot 3 \dots \delta}$$

according to the notation of *Euler*.* For numerical computations we then obtain from (6)

$$\left. \begin{aligned} \Sigma_1 &= A_1 + 2 A_2 B_0 + 3 A_3 B_0^2 + 4 A_4 B_0^3 + \dots \\ \Sigma_2 &= A_2 + 3 A_3 B_0 + 6 A_4 B_0^2 + \dots \\ \Sigma_3 &= A_3 + 4 A_4 B_0 + \dots \\ \Sigma_4 &= A_4 + \dots \end{aligned} \right\} \quad (7)$$

Thus the conditions (1) to (5) present themselves in the more perspicuous forms

$$\left. \begin{aligned} \Sigma_0 &= 0 \\ B_1 \Sigma_1 &= 1 \\ B_2 \Sigma_1 + B_1^2 \Sigma_2 &= 0 \\ B_3 \Sigma_1 + 2 B_1 B_2 \Sigma_2 + B_1^3 \Sigma_3 &= 0 \\ B_4 \Sigma_1 + 2 (B_1 B_3 + \frac{1}{2} B_2^2) \Sigma_2 + 3 B_1^2 B_2 \Sigma_3 + B_1^4 \Sigma_4 &= 0. \end{aligned} \right\} \quad (8)$$

The law of these series being evidenced from inspection, we deduce the next following conditions :

$$\begin{aligned} B_5 \Sigma_1 + 2 (B_1 B_4 + B_2 B_3) \Sigma_2 + 3 (B_1^2 B_3 + B_2^2 B_1) \Sigma_3 + 4 B_1^3 B_2 \Sigma_4 + \\ B_1^5 \Sigma_5 &= 0 \\ B_6 \Sigma_1 + \pi(2) (B_1 B_5 + B_2 B_4 + \frac{B_3^2}{\pi(2)}) \Sigma_2 + \pi(3) (B_1 B_2 B_3 + \frac{B_1^2}{\pi(2)} B_4 + \frac{B_2^3}{\pi(3)}) \\ \times \Sigma_3 + \pi(4) \left(\frac{B_1^3}{\pi(3)} B_3 + \frac{B_1^2}{\pi(2)} \frac{B_2^2}{\pi(2)} \right) \Sigma_4 + 5 B_1^4 B_2 \Sigma_5 + B_1^6 \Sigma_6 &= 0, \end{aligned}$$

and in the general form

$B_\mu \Sigma_1 + \pi(2)(B_1 B_{\mu-1} + \dots) \Sigma_2 + \pi(3)(B_1 \dots) \Sigma_3 + \dots + B_1^\mu \Sigma_\mu = 0$, each term with the sign Σ_ν having the factor $\pi(\nu)$ and each B^σ having the denominator $\pi(\sigma)$. The factors of Σ_ν are always ν in number, and the sum of their indices μ .

There is no difficulty in solving the equations (8), except the first,

* *Acta Petropolitana*, V. 1, p. 89. Though his notation is not much used in American text-books, it is found very handy in operating on series.

which is of the m th degree and will be considered presently. The other equations give the following solutions :

$$\left. \begin{aligned} B_1 &= + \frac{1}{\Sigma_1} \\ B_2 &= - \frac{1}{\Sigma_1^3} \Sigma_2 \\ B_3 &= + \frac{1}{\Sigma_1^5} (2 \Sigma_2^2 - \Sigma_1 \Sigma_3) \\ B_4 &= - \frac{1}{\Sigma_1^7} (5 \Sigma_2^3 - 5 \Sigma_1 \Sigma_2 \Sigma_3 + \Sigma_1^2 \Sigma_4), \text{ etc.} \end{aligned} \right\} (9)$$

The formulas show, first, that in general we shall have $\mu = \infty$, and secondly, that the series of the coefficients B decreases the faster the larger Σ_1 is. For the quotient $B_{\mu+1} \div B_\mu$ is of the same order as $1 \div \Sigma_1$. Hence a few terms will suffice to compute y as often as the coefficient A_1 is large in comparison to the following coefficients.

Now as to the condition

$$\Sigma_0 = \sum_{r=0}^{r=m} A_r B_0 B_0^r = 0,$$

it is evident that its *exact* solution is impossible as often as $m > 4$, except in one case, viz.: when $A_0 = 0$, in which we have also $B_0 = 0$. The approximate solution of the above equation by development will be explained in Part II.

§3. In the special case $A_0 = 0$ we have $B_0 = 0$, because x and y are zero at the same time. Consequently we have $\Sigma_\delta = A_\delta$ and the formulas (8) may be written in the form

$$\left. \begin{aligned} B_0 &= A_0 = 0 \\ B_1 A_1 &= 1 \\ B_2 A_1 + B_1^2 A_2 &= 0 \\ B_3 A_1 + 2 B_1 B_2 A_2 + B_1^3 A_3 &= 0 \\ B_4 A_1 + 2 (B_1 B_3 + \frac{1}{2} B_2^2) A_2 + 3 B_1^2 B_2 A_3 + B_1^4 A_4 &= 0 \\ B_5 A_1 + 2 (B_1 B_4 + B_2 B_3) A_2 + 3 (B_1^2 B_3 + B_2^2 B_1) A_3 + \\ &\quad 4 B_1^3 B_2 B_4 + B_1^5 A_5 = 0 \end{aligned} \right\} (8')$$

etc., their solutions being

$$\left. \begin{aligned} B_1 &= + \frac{1}{A_1} \\ B_2 &= - \frac{1}{A_1^3} A_2 \\ B_3 &= + \frac{1}{A_1^5} (2 A_2^2 - A_1 A_3) \\ B_4 &= - \frac{1}{A_1^7} (5 A_2^3 - 5 A_1 A_2 A_3 + A_1^2 A_4), \text{ etc.} \end{aligned} \right\} (9')$$

PART II.

§4. When A_0 is not zero, the given series may be written in the form

$$x - A_0 = \sum_{r=1}^{r=m} A_r y^r \quad (10)$$

Here we have exactly the case of §3 and the formulas (9') will at once give the coefficients of the series

$$y = \sum_{\delta=0}^{\delta=\mu} B_{\delta} (\mathfrak{X} - A_0)^{\delta}. \quad (10')$$

§5. When it is required to have y developed into a series of ascending powers of $\mathfrak{X}$ itself, we may proceed in the following way. Let the given series be written in two different ways,

$$\mathfrak{X} = \sum_{r=0}^{r=m} A_r y^r \text{ and } \mathfrak{X} - A_0 = \sum_{r=1}^{r=m} A_r y^r,$$

and consequently also the reversed series

$$y = \sum_{\delta=0}^{\delta=\infty} B_{\delta} \mathfrak{X}^{\delta} \text{ and } y = \sum_{\delta=0}^{\delta=\infty} C_{\delta} (\mathfrak{X} - A_0)^{\delta}. \quad (11)$$

The values of the C are given by the formulas (9'), provided that we write C instead of B , as has been explained in §4, while the coefficients B are still unknown. Developing $(\mathfrak{X} - A_0)^{\delta}$ by the Newtonian formula, we get

$$(\mathfrak{X} - A_0)^{\delta} = (-1)^{\delta} \sum_{\lambda=0}^{\lambda=\infty} (-1)^{\lambda} \binom{\delta}{\lambda} A_0^{\delta-\lambda} \mathfrak{X}^{\lambda};$$

and equating the two series (11), we obtain

$$\sum_{\delta=0}^{\delta=\infty} B_{\delta} \mathfrak{X}^{\delta} = \sum_{\lambda=0}^{\lambda=\infty} (-1)^{\lambda} \mathfrak{X}^{\lambda} \sum_{\delta=0}^{\delta=\infty} (-1)^{\delta} C_{\delta} \binom{\delta}{\lambda} A_0^{\delta-\lambda};$$

and finally by the theorem of *Indeterminate Coefficients*,

$$B_{\lambda} = (-1)^{\lambda} \sum_{\delta=\lambda}^{\delta=\infty} (-1)^{\delta} \binom{\delta}{\lambda} C_{\delta} A_0^{\delta-\lambda} \quad (12)$$

This formula may be transformed, by changing the index $\delta = \lambda + r$, thus :

$$B_{\lambda} = (-1)^{\lambda} \sum_{r=0}^{r=\infty} (-1)^{\lambda+r} \binom{\lambda+r}{\lambda} C_{\lambda+r} A_0^r, \quad (12')$$

where instead of $\binom{\lambda+r}{\lambda}$ we may write $\binom{\lambda+r}{r}$. The convergence of the series (12) will depend upon the coefficients C and must be examined in each special case; in general we can state that it always converges, when we have

$$\lim_{\alpha=\infty} \frac{C_{\delta+1}}{C_{\delta}} < \pm \frac{1}{A_0}$$

Example.—Let it be required to reverse the series

$$\mathfrak{X} = 1 + \frac{1}{1} y + \frac{1}{2} y^2 + \frac{1}{3} y^3 + \dots$$

Here is $A_{\delta} = \frac{1}{\delta}$ ($\delta > 0$) and $A_0 = 1$, hence we obtain from (9'), writing C instead of B ,

$$C_0=0, C_1=1, C_2=-\frac{1}{\pi(2)}, C_3=+\frac{1}{\pi(3)}, C_4=-\frac{1}{\pi(4)}, \text{ etc.,}$$

and in general (except C_0)

$$C_\delta = (-1)^{\delta+1} \frac{1}{\pi(\delta)} \text{ or } C_{\lambda+r} = \frac{(-1)^{\lambda+r+1}}{\pi(\lambda+r)}.$$

This value substituted in (12') gives for $\lambda > 1$

$$B_\lambda = (-1)^{\lambda+1} \sum_{r=0}^{\infty} \frac{1}{\lambda+r} \frac{1}{\pi(\lambda+r)} = \frac{(-1)^{\lambda+1}}{\pi(\lambda)} \sum_{r=0}^{\infty} \frac{1}{\pi(r)} = \frac{(-1)^{\lambda+1}}{\pi(\lambda)} \times e,$$

and for $\lambda = 0$, since $C_0 = 0$,

$$B_0 = - \sum_{r=1}^{\infty} \frac{1}{\pi(r)} = 1 - \sum_{r=0}^{\infty} \frac{1}{\pi(r)} = 1 - e;$$

where $e = 2.718281828 +$ is the base of the *Nepierian* system of logarithms.

Substituting these values into the first of the formulas (11) we obtain

$$y = \sum_{\lambda=0}^{\infty} B_\lambda x^\lambda = 1 - e + e \sum_{\lambda=1}^{\infty} \frac{(-1)^{\lambda+1}}{\pi(\lambda)} x^\lambda.$$

Consequently the given series

$$x = 1 + \frac{y}{1} + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \dots = 1 + \sum_{r=1}^{\infty} \frac{y^r}{r}$$

is reversed in the following way :

$$\begin{aligned} y &= 1 - e \left(1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \dots \right) \\ &= 1 - e \sum_{r=0}^{\infty} (-1)^r \frac{x^r}{\pi(r)}. \end{aligned}$$

This last formula may be tested in the following way. The given series requires that we have at the same time $x = 1$ and $y = 0$, consequently the reversed series requires the identity

$$1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \dots = \frac{e-1}{e},$$

which may be verified without difficulty.

PART III.

The equations (10) and (10') imply the *approximate solution of algebraic equations*. Putting $A_0 = 0$ and assuming for x any constant quantity, say a , we may write these equations in the following way :

$$a = \sum_{r=1}^m A_r y^r, \text{ solution, } y = \sum_{\delta=0}^{\infty} B_\delta a^\delta \quad (13)$$

The coefficients B are determined by the equations (8) and (9). We do not say (8') and (9'), because the condition $B_0 = 0$ is not by necessity fulfilled in this case, although we have $A_0 = 0$. While in §1 we have stated that the reversion of the series admits of but one root of the equation (1), since there is but one way of developing y into a series of ascending powers of x , we now have to say, that all the m roots of

the equation (1) are to be considered, since our equation (13) of the m th degree admits of m solutions. And indeed, equation (13) being no more an identity, we cannot say, as we did in §3, that a and y will be zero at the same time.

Consequently the condition (1) is to be taken in its full extent and, since $A_0 = 0$, may be written this way:

$$\sum_{r=1}^{r=m} A_r B_0^r = 0 \quad (14)$$

This equation is at once resolved into the following two:

$$B_0 = 0 \text{ and } \sum_{r=1}^{r=m} A_r B_0^{r-1} = 0, \quad (14')$$

thus showing that *the solution of the equation of the m th degree is made dependent on the solution of an equation of the $(m-1)$ th degree*, as has been remarked also by Prof. Schlömilch on page 26 of his article referred to. For each root B_0 the formulas (7) and (8) will furnish a different set of coefficients $B_1 B_2 \dots$ and consequently a different value of y , and the formulas (9') give at once the value of y for the root $B_0 = 0$:

$$y = + \frac{1}{A_1} a - \frac{1}{A_1^3} A_2 a^2 + \frac{1}{A_1^5} (2 A_2^2 - A_1 A_3) a^3 - \frac{1}{A_1^7} (5 A_2^3 - 5 A_1 A_2 A_3 + A_1^2 A_4) a^4 + \dots \quad (15)$$

As we have already noted in §2, this method is applicable especially to such series in which the first coefficient A_1 is large in comparison to the following ones.

First example.—Let the given equation be

$$y^2 + 10 y + 1 = 0.$$

Here we have $A_2 = 1$, $A_1 = 10$, $a = -1$, and from (14') we get the two conditions $B_0 = 0$ and $B_0 = -10$. By the first we obtain from (15)

$$y_1 = -\frac{1}{10} - \frac{1}{10^3} - \frac{2}{10^5} - \frac{5}{10^7} - \dots = -.1010205 \dots$$

By the second condition $B_0 = -10$, we compute from (7)

$$\Sigma_1 = -10, \Sigma_2 = 1, \Sigma_3 = 0, \text{ etc.};$$

and by means of these values from (8) or (9)

$$B_1 = -\frac{1}{10}, B_2 = \frac{1}{10^3}, B_3 = -\frac{2}{10^5}, B_4 = \frac{5}{10^7}, \text{ etc.};$$

and finally we have from (13)

$$y_2 = -10 + \frac{1}{10} - \frac{1}{10^3} + \frac{2}{10^5} - \frac{5}{10^7} + \dots = -9.8989795 \dots$$

A proof of this calculation is found in that $y_1 + y_2$ is equal to the negative coefficient of y .

Second example.—Let it be required to solve the equation of the fourth degree

$$y^4 - 4 y^3 - 25 y^2 + 100 y + 1 = 0.$$

Here we have $A_4 = 1$, $A_3 = -4$, $A_2 = -25$, $A_1 = 100$, $a = -1$.

The equations (14) are now

$$B_0 = 0 \text{ and } B_0^3 - 4 B_0^2 - 25 B_0 + 100 = 0.$$

The latter admits of being resolved in the following way :

$$B_0^3 - 4 B_0^2 - 25 B_0 + 100 = (B_0^2 - 25) (B_0 - 4),$$

and thus we obtain for B_0 the following numerical values 0, -5, +4, +5.

1. By the first we obtain from (15)

$$y_1 = -\frac{1}{100} + \frac{25}{100^3} - \frac{1650}{100^5} + \frac{118125}{100^7} - \dots \\ = -.009,975,163,8 \dots$$

2. By the second we compute from (7)

$$\Sigma_1 = -450, \Sigma_2 = +185, \Sigma_3 = -24, \Sigma_4 = 1, \Sigma_5 = 0, \text{ etc.,}$$

and consequently from (8) or (9)

$$B_1 = -\frac{1}{450}, B_2 = +\frac{185}{450^3}, B_3 = -\frac{57650}{450^5}, \text{ etc.,}$$

and finally from (13)

$$y_2 = -5 + \frac{1}{450} + \frac{185}{450^3} + \frac{57650}{450^5} + \dots \\ = -4.997,775,744,5 \dots$$

3. In the third case $B_0 = +4$ we find from (7)

$$\Sigma_1 = -36, \Sigma_2 = +23, \Sigma_3 = +12, \Sigma_4 = 1, \Sigma_5 = 0, \text{ etc.,}$$

and from (9)

$$B_1 = -\frac{1}{36}, B_2 = \frac{23}{36^3}, B_3 = -\frac{1490}{36^5}, B_4 = \frac{80707}{36^7}, \text{ etc.;}$$

hence from (13)

$$y_3 = 4 + \frac{1}{36} + \frac{23}{36^3} + \frac{1490}{36^5} + \frac{80707}{36^7} + \dots \\ = 4.028,296,8 \dots$$

4. In the fourth case $B_0 = +5$ we have from (7)

$$\Sigma_1 = +50, \Sigma_2 = +65, \Sigma_3 = +16, \Sigma_4 = 1, \Sigma_5 = 0, \text{ etc.,}$$

and from (9)

$$B_1 = \frac{1}{50}, B_2 = -\frac{65}{50^3}, B_3 = \frac{7650}{50^5}, B_4 = -\frac{1115625}{50^7}, \text{ etc.;}$$

consequently :

$$y_4 = 5 - \frac{1}{50} - \frac{65}{50^3} - \frac{7650}{50^5} - \frac{1115625}{50^7} - \dots \\ = +4.979,453,9 \dots$$

A proof of the work is found in the sum of the four roots,

$$\begin{aligned} y_1 &= -0.009,975,163,8 \\ y_2 &= -4.997,775,744,5 \\ y_3 &= +4.028,296,8 \\ y_4 &= +4.979,453,9 \\ \hline &+ 4.000,000, \end{aligned}$$

inclusive of the sixth decimal place being equal to the negative coefficient of y^3 .